

# ON A RESULT OF GÁBOR CZÉDLI CONCERNING CONGRUENCE LATTICES OF PLANAR SEMIMODULAR LATTICES

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**ABSTRACT.** A planar semimodular lattice is *slim* if it does not contain  $M_3$  as a sublattice. An SPS lattice is a slim, planar, semimodular lattice.

Congruence lattices of SPS lattices satisfy a number of properties. It was conjectured that these properties characterize them. A recent result of Gábor Czédli proves that there is an eight element (planar) distributive lattice having all these properties that cannot be represented as the congruence lattice of an SPS lattice.

We provide a new proof.

## 1. INTRODUCTION

Let  $L$  be a planar semimodular lattice. If  $L$  is a *slim* lattice (G. Grätzer and E. Knapp [12]), that is, it contains no  $M_3$  sublattice, we call it an SPS lattice (Slim, Planar, Semimodular).

G. Grätzer, H. Lakser, and E. T. Schmidt [10] prove that every finite distributive lattice can be represented as the congruence lattice of a planar semidistributive lattice. The proof heavily relies on  $M_3$  sublattices.

I raised in [8] the problem of characterizing congruence lattices of SPS lattices and I proved in [9] the following necessary condition:

**Theorem 1.** *Let  $L$  be an SPS lattice. Then the order of join-irreducible congruences of  $L$  satisfies the following condition:*

(CC1) *an element is covered by at most two elements.*

G. Czédli observed that it follows from G. Czédli [2, Lemma 2.2] that the order of join-irreducible congruences of an SPS lattice  $L$  also satisfies the following condition:

(CC2) *for any nonmaximal element  $a$ , there are (at least) two distinct maximal elements  $m_1$  and  $m_2$  above  $a$ .*

G. Czédli [2] proved (using his Trajectory Coloring Theorem [1], which has a 30 page proof) that the converse does not hold.

**Theorem 2.** *The eight element distributive lattice  $D_8$  of Figure 1 cannot be represented as the congruence lattice of an SPS lattice  $L$ .*

Note that the order,  $J(D_8)$ , of join-irreducible elements of  $D_8$ —see Figure 1—satisfies conditions (CC1) and (CC2).

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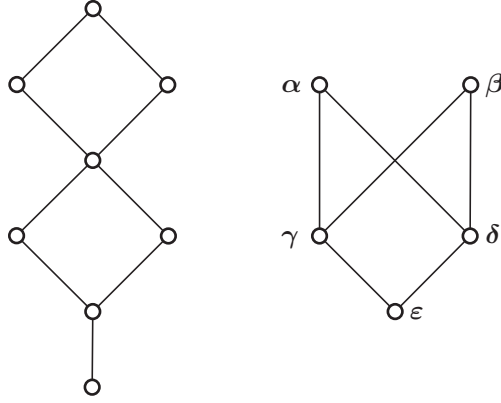
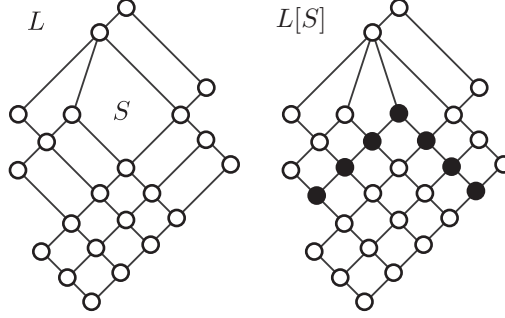
FIGURE 1. The lattice  $D_8$  and the order  $P = J(D_8)$ 

FIGURE 2. The fork construction, two steps

In this note, we provide a new proof of Theorem 2 that does not utilize the Trajectory Coloring Theorem. However, the new proof utilizes some nontrivial results, see Theorems 3 and 4.

For the basic concepts and notation, see G. Grätzer [6]. For an overview of SPS lattices, see G. Czédli and G. Grätzer [3] and G. Grätzer [7], Chapters 3 and 4 of G. Grätzer and F. Wehrung eds. [14].

## 2. FORKS

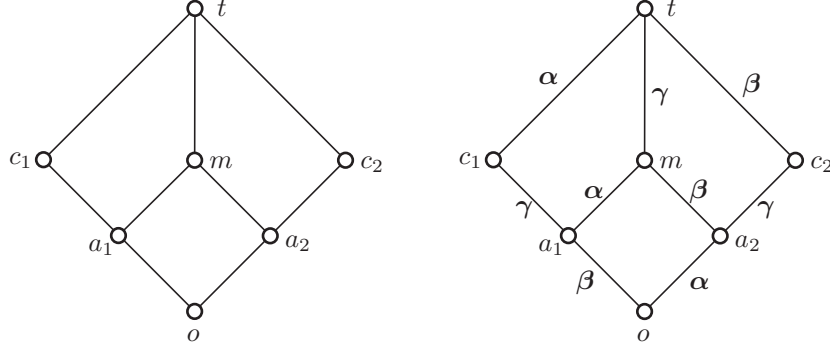
We need some basic definitions and results.

Let  $L$  be an SPS lattice. A 4-cell in  $L$  is a covering  $C_2^2$  with no interior element.

Let  $S$  be a 4-cell of  $L$ . We construct a lattice extension  $L[S]$  of  $L$  as follows.

Firstly, we replace  $S$  by a copy of  $N_7$ , the lattice of Figure 3, introducing three new element.

Secondly, we do a series of steps—each step introducing one new element: if there is a chain  $u \prec v \prec w$  such that  $v$  is a new element but  $u$  and  $w$  are not, and  $T = \{u \wedge x, x, u, w = x \vee u\}$  is a 4-cell in the original lattice  $L$ , see Figure 2, then we insert a new element  $y$  such that  $x \wedge u \prec y \prec x \vee u$  and  $y \prec v$ . Figure 2 shows some steps of the construction.

FIGURE 3. The lattices  $N_7$  and  $L_1$ 

Let  $L[S]$  denote the lattice we obtain when the procedure terminates. We say that  $L[S]$  is obtained from  $L$  by *inserting a fork* at the 4-cell  $S$ .

Let  $L$  be a planar lattice. A *left corner* (resp., *right corner*) of  $L$  is a doubly irreducible element in  $L - \{0, 1\}$  on the left (resp., right) boundary of  $L$ . As in G. Grätzer and E. Knapp [13], we define a *rectangular lattice*  $L$  as a planar semi-modular lattice which has exactly one left corner,  $lc(L)$ , and exactly one right corner,  $rc(L)$ , and they are complementary. A rectangular lattice  $L$  is a *patch lattice* (G. Czédli and E. T. Schmidt [4] and [5]) if  $lc(L)$  and  $rc(L)$  are dual atoms.

We need the following result of G. Czédli and E. T. Schmidt [4]:

**Theorem 3** (Structure Theorem for SPS Lattices). *An SPS lattice  $L$  can be obtained from a planar distributive lattice  $D$  by inserting forks.*

We call the planar distributive lattice  $D$  the *grid* of the SPS lattice  $L$ .

In an SPS lattice  $L$ , we call the covering square of  $S = \{o, c_l, c_r, t\}$  a *tight square*, if  $t$  covers exactly two elements, namely,  $c_l$  and  $c_r$ , in  $L$ ; otherwise,  $S$  is a *wide square*.

The following results are from G. Grätzer [8] (the notation  $o$ ,  $a_l$ ,  $a_r$ ,  $t$ , and  $m$  is from Figure 3, denoting the elements of the inserted  $N_7$ ).

**Theorem 4.** *Let  $L$  be an SPS lattice and let  $S$  be a covering square of  $L$ . If  $S$  is a*

- (i) *wide square, then the congruence  $\gamma(S) = \text{con}_{L[S]}(m, t)$  of  $L[S]$  is generated by a congruence of  $L$ .*
- (ii) *tight square, then  $L[S]$  has exactly one join-irreducible congruence, namely  $\gamma(S) = \text{con}_{L[S]}(m, t)$ , that is not generated by a congruence of  $L$ .*

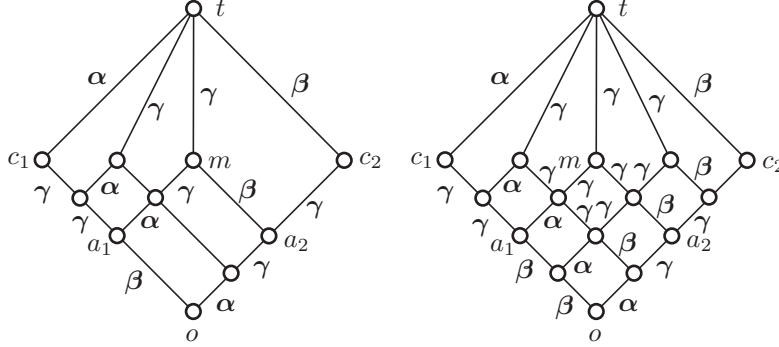
### 3. PROOF OF THEOREM 2

Let  $L$  be a finite SPS lattice whose join-irreducible congruences form an order  $P$  as in Figure 1. By G. Grätzer and E. Knapp [13],  $L$  has a congruence-preserving extension to a rectangular lattice, so we can assume that  $L$  is a rectangular lattice.

*Case 1:  $L$  is a patch lattice.*

By the Structure Theorem for SPS Lattices, we can obtain  $L$  from  $C_2^2$  by inserting forks. Let  $C_2^2 = L_0, L_1, \dots, L_n = L$  be a sequence of fork insertions from  $C_2^2$  to  $L$ .

There is only one way to insert a fork into  $C_2^2$ ; so  $L_1$  is the lattice of Figure 3. There is one more join-irreducible congruence in  $L_1$ , the congruence  $\gamma$ . We “color” the diagram of  $L_1$  with the join-irreducible congruences generated by the edges.

FIGURE 4. The lattices  $L_2$  and  $L_3$  with no new congruence

To get  $L_2$ , we pick a covering square  $S_1$  in  $L_1$  and insert a fork into  $L_1$  at  $S_1$ . If the top element of  $S_1$  is  $t$ , then  $S_1$  is a wide square, so we get no new congruence by Theorem 4(i), see Figure 4. The next step, again with no new congruence is shown in the same figure. Note that all covering squares of the lattices  $L_1, L_2, L_3$  satisfy the following condition:

- (Col) all covering squares are colored by  $\gamma$  by itself or with  $\alpha$  or with  $\beta$  with one exception: the bottom covering square is colored by  $\alpha$  and  $\beta$ .

We proceed thus in  $k - 1$  steps to get  $L_k$ , where  $k \geq 1$  is the largest number with the property that the number of join-irreducible congruences do not change from  $L_{k-1}$  to  $L_k$ . Note that  $k < n$  because  $L_n = L$  has more join-irreducible congruences. Clearly,  $L_k$  also satisfies condition (Col).

We proceed to  $L_{k+1}$  by inserting a fork into a covering square  $S$  of  $L_k$ . There are five cases:

- (i) the top element of  $S$  is  $t$ ;
- (ii) the top element of  $S$  is not  $t$  and  $S$  is “monochromatic”, colored by  $\gamma$ ;
- (iii) the top element of  $S$  is not  $t$  and  $S$  is colored by  $\{\gamma, \alpha\}$ ;
- (iv) the top element of  $S$  is not  $t$  and  $S$  is colored by  $\{\gamma, \beta\}$ ;
- (v)  $S$  is the bottom covering square colored by  $\{\alpha, \beta\}$ .

Case (i) cannot happen, it would contradict Theorem 4(i) and the definition of  $k$ .

If Case (ii) holds,  $S$  is tight, so by Theorem 4(ii), we add a join-irreducible congruence  $\gamma' < \gamma$ . By Figure 1, we must have  $\gamma' = \varepsilon$ . This is a contradiction because no fork insertion can add an element  $\delta$  between two existing elements.

Case (iii) proceeds the same way as Case (ii).

Case (iv) is symmetric to Case (iii).

So we are left with Case (v). In this case, the top element of  $S$  is not  $t$ , so  $S$  is tight. By Theorem 4(ii), we get a new join-irreducible congruence  $\delta$  in  $L_{k+1}$  satisfying that  $\delta < \alpha$  and  $\delta < \beta$ , see Figure 5.

Let  $k + 1 \leq m < n$  be the largest integer so that the number of join-irreducible congruences does not change from  $L_{k+1}$  to  $L_m$ —we insert forks into wide squares. Then the lattices  $L_{k+1}, \dots, L_m$  share the following property of  $L_{k+1}$ :

- (P) there are only “monochromatic” squares and  $\{\alpha, \beta\}$ ,  $\{\alpha, \gamma\}$ ,  $\{\alpha, \delta\}$ ,  $\{\beta, \gamma\}$ ,  $\{\beta, \delta\}$  squares, but there is no  $\{\gamma, \delta\}$  square.

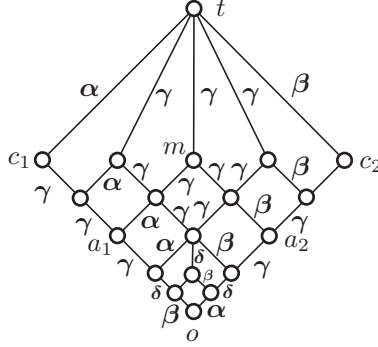
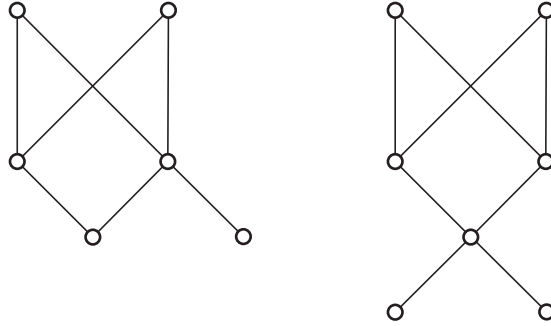
FIGURE 5. The lattice  $L_{k+1}$  for case (v)

FIGURE 6. Larger nonrepresentable orders

We obtain the last join-irreducible congruence,  $\varepsilon$ , in  $L_{m+1}$ . By property (P), we cannot have  $\varepsilon \prec \gamma$  and  $\varepsilon \prec \delta$ .

*Case 2:*  $L$  is not a patch lattice. It follows that the grid  $D$  is not a covering square. The lattice  $D$  has three join-irreducible congruences forming an antichain of 3 elements. But the order of Figure 1 has only one- and two-element antichains.

This completes the proof of Theorem 2.

The two cases of the proof can easily be unified. If  $L$  is not a patch lattice, then there are a number of covering squares we can start with. Say, we start with the covering square  $S$  colored by  $\alpha$  and  $\beta$ . We proceed as in Case 1, except that there are more colors. The additional colours do not change the fact that there are no  $\{\gamma, \delta\}$  squares.

#### 4. LARGER NONREPRESENTABLE ORDERS

The proof of Theorem 2 can be made to work under related assumptions. For instance, the orders in Figure 6 are not representable.

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